EXACT SOLUTIONS OF SOME PROBLEMS CONCERNED WITH OSCILLATIONS OF FLUID CONTAINED IN AN ELASTIC MOMENTLESS SHELL

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This paper presents a study of small, steady free oscillations of a system consisting of an elastic shell filled with an ideal incompressible fluid.

To describe the motion of the shell differential equations of the momentless [membrane] theory are used. The applicability of this theory for computing of frequencies which are not too high and of the modes of oscillation was already established in [1]. The motion of the fluid is taken as potential one.

Exact solutions of the problems of free oscillations for the following types of shell fully filled with fluid were obtained: a cylindrical shell with one or two rigid ends, a closed spherical, and a half-spherical one.

1. Equations of motion of the cylindrical shell are

$$\left[\frac{\partial^2}{\partial\theta^2}+\frac{1-\sigma}{2}\frac{\partial^2}{\partial\varphi^2}-\frac{(1-\sigma^2)\rho R^2}{E}\frac{\partial^2}{\partial t^2}\right]u+\frac{1+\sigma}{2}\frac{\partial^2 v}{\partial\theta\partial\varphi}-\sigma\frac{\partial w}{\partial\theta}=0$$

$$\frac{1+\sigma}{2}\frac{\partial^2 u}{\partial\theta\partial\varphi}+\left[\frac{1-\sigma}{2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial\varphi^2}-\frac{(1-\sigma^2)\rho R^2}{E}\frac{\partial^2}{\partial t^2}\right]v-\frac{\partial w}{\partial\varphi}=0 \qquad (1.1)$$

$$-\sigma\frac{\partial u}{\partial\theta}-\frac{\partial v}{\partial\varphi}+\left[1+\frac{(1-\sigma^2)\rho R^2}{E}\frac{\partial^2}{\partial t^2}\right]w+\frac{(1-\sigma^2)R^2}{2Eh}(p)_{\xi=1}=0$$

where apart from the usual symbols, $2h$ is the thickness of the shell, ρ is the density of material, $(p)_{\xi=1}$ is the hydrodynamic pressure of the fluid on the sides of the shell.

1. Let us consider the free oscillations of the cylindrical shell with a rigid bottom, fully filled with fluid.

The volume of fluid contained in the cylindrical system of coordinates is limited by the surface of the shell $\xi = 1$ and the bottoms $\theta = 0$, $\theta = 2H / R$ (where R and $2H$ respectively, are the radius and length of the shell).

The potential of the velocities of particles of fluid, satisfying the Laplace equation and the condition of regularity in the fluid can be written as

$$\Phi_{nm} = A_{nm} I_n(\nu_m\xi)\cos\nu_m\theta\cos n\varphi\sin\omega_{nm}t \qquad (n = 0, 1, 2, \ldots) \qquad (1.2)$$

where $I_n(x)$ is a cylindrical function of an imaginary argument.

Taking

$$\nu_m = m\pi R / 2H \quad (m = 1, 2, 3, \ldots) \qquad (1.3)$$

we can satisfy the condition of rigid, fixed bottoms

$$\partial\Phi / \partial\theta = 0 \qquad \text{for } \theta = 0,\ \theta = 2H / R$$

The hydrodynamic pressure of fluid, entering (1.1) is determined by formula

$$(p)_{\xi=1} = -\rho_* (\partial\Phi/\partial t)_{\xi=1} \tag{1.4}$$

where ρ_* is the density of the fluid.

The condition of equal velocities of particles of the fluid and of the shell in the direction of the normal to its middle surface leads to (1.5), where the positive value of w corresponds to displacement in the direction of the internal normal

$$R\,(\partial w / \partial t) + (\partial\Phi / \partial\xi)_{\xi=1} = 0 \tag{1.5}$$

The system (1.1), bearing in mind (1.2),(1.4) and (1.5), admits exact solution

$$\begin{aligned} u_{nm} &= C_{nm}^{(1)} \sin \nu_m\theta \cos n\varphi \cos \omega_{nm} t \\ v_{nm} &= C_{nm}^{(2)} \cos \nu_m\theta \sin n\varphi \cos \omega_{nm} t \\ w_{nm} &= C_{nm}^{(3)} \cos \nu_m\theta \cos n\varphi \cos \omega_{nm} t \end{aligned} \tag{1.6}$$

The constants in (1.2) and (1.6) can be determined up to an arbitrary multiple C_{nm} by formulas

$$\begin{aligned} C_{nm}^{(1)} &= [\sigma\nu_m^2 - n^2 - 2\sigma(1+\sigma)\lambda_{nm}^2]\,\nu_m C_{nm} \\ C_{nm}^{(2)} &= [n^2 + (2+\sigma)\nu_m^2 - 2(1+\sigma)\lambda_{nm}^2]\,nC_{nm} \\ C_{nm}^{(3)} &= [(\nu_m^2 + n^2)^2 + 2(1+\sigma)(1-\sigma^2)\lambda_{nm}^4 - (1+\sigma)(3-\sigma)(\nu_m^2+n^2)\lambda_{nm}^2]\,C_{nm} \end{aligned} \tag{1.7}$$

$$\begin{aligned} [\nu_m I_{n-1}(\nu_m) - nI_n(\nu_m)]\,A_{nm} = [(\nu_m^2 + n^2)^2 + 2(1+\sigma)(1-\sigma^2)\lambda_{nm}^4 - \\ -(1+\sigma)(3-\sigma)(\nu_m^2+n^2)\lambda_{nm}^2]\,R\omega_{nm}C_{nm} \end{aligned}$$

The dimensionless parameter of the frequency of oscillation $\lambda_{mn}^2 = \rho R^2\omega_{mn}^2/E$ is determined by equation

$$\begin{aligned} [\nu_m I_{n-1}(\nu_m) - nI_n(\nu_m)]\,\{\nu_m^4 - 2(1+\sigma)(1-\sigma^2)\lambda_{nm}^6 + 2(1+\sigma)[1+(3-\sigma)\times \\ \times(\nu_m^2+n^2)]\lambda_{nm}^4 - [n^2+(3+2\sigma)\nu_m^2 + (\nu_m^2+n^2)^2]\lambda_{nm}^2\}\,c - I_n(\nu_m)\,[(\nu_m^2+n^2)^2 + \\ + 2(1+\sigma)(1-\sigma^2)\lambda_{nm}^4 - (1+\sigma)(3-\sigma)(\nu_m^2+n^2)\lambda_{nm}^2]\,\lambda_{nm}^2 = 0 \end{aligned} \tag{1.8}$$

where $c = 2h\rho / \rho_* R$. The momentless boundary conditions for the shell corresponding to (1.6) are

$$u = S_1 = 0 \qquad \text{for } \theta = 0,\ \theta = 2H / R$$

where S_1 is the shear force in the transverse cross section of the shell. Neglecting the inertia forces of the shell, we have $\lambda_{nm} = 0$ $(\omega_{nm} \neq 0)$ in (1.7) and the frequency equation (1.8) becomes

$$[\nu_m I_{n-1}(\nu_m) - nI_n(\nu_m)]\,c\nu_m^4 - \lambda_{nm}^2 I_n(\nu_m)(\nu_m^2+n^2)^2 = 0 \tag{1.9}$$

For fixed n and m Eq. (1.8) gives three values for λ_{nm}^2; however, two of them must be neglected, because they are far too great (the hypothesis of incompressibility of the fluid and momentless theory of the shell would become unjustified).

We see that the modes of oscillation (1.6) for the even values of m are symmetrical and for the odd values are anti-symmetrical relative to the mean section $\theta = H / R$.

2. Let us now consider the free oscillations of the cylindrical shell fully filled with fluid, but with only one rigid bottom. The length of the shell is H, acceleration of the

gravity field is g and is directed opposite to the axis θ. It is obvious that the above solution also satisfies this case if in (1.3) we accept only the odd values of m. Boundary conditions for $\theta = 0$ remain the same as in the previous problem. The conditions related to the other section are:-

$$v = T_1 = p = 0, \qquad \text{for } \theta = H / R$$

where T_1 is the longitudinal force in the cross section of the shell.

If the problem is solved for the condition

$$R\,(\partial^2\Phi / \partial t^2) + g\,(\partial\Phi / \partial\theta) = 0 \qquad \text{for } \theta = H / R \tag{1.10}$$

which is more severe than $(p)_{\theta=H/R} = 0$, we get the relation

$$\lambda_{nm}^2 + \delta\nu_m \operatorname{tg}(\nu_m H / R) = 0 \tag{1.11}$$

which permits, using (1.8), to determine ν_m and $\lambda_{nm}{}^2$. Here $\delta = \rho g R / E$. For constructions which are met in usual practice we find a strong inequality $\delta \ll \lambda^2$. Then from (1.11) we can find that ν_m with a great degree of accuracy is identical with (1.3) for odd values of m and therefore the different conditions quoted for the free surface of fluid produce practically the same results.

For realization of the nontangential boundary conditions, attached to function w and its derivatives, we have to add to the momentless solution the solutions with great variability, which have the character of a boundary effect [1].

In many studies concerned with free oscillations of elastic shells with fluid, the forces of inertia of the shell are disregarded. In the problem under consideration, connected with this assumption, the error can be easily evaluated by comparing the frequencies obtained by (1.8) and (1.9). Frequencies found by (1.9), as should be expected, are too large and the value of the error increases with the frequency and substantially depends on the values c and H / R. In Fig.1 this relation for $n = 0$ is given for the first two frequencies of the oscillation ($m = 1$ and 3). The axis of the ordinates corresponds to the percentage error in determining λ^2 by (1.9) instead of (1.8). It was assumed that $\sigma = 0.3$.

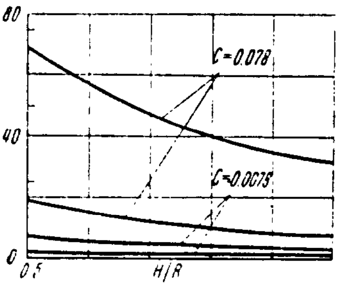

Fig. 1.

In the case of axially symmetrical free oscillations ($n = 0$) this problem was solved numerically with the help of an electronic computer. The thickness of the wall of the cylindrical shell was taken as a linear function of the coordinate θ. It was found that the variation of constants a and b in the boundary case is

$$au + bT_1 = 0 \qquad \text{for } \theta = H / R \tag{1.12}$$

and on the spectrum of frequencies this was apparent only when the forces of inertia of the shell play a substantial part. Therefore, if the difference of frequencies of free oscillations, evaluated by (1.8) and (1.9) is small, the solution obtained can be considered to be a sufficiently close approximation for the solution of the free oscillations of the shell with boundary conditions (1.12) and for any values of the constants a and b (e.g. for partial filling of the cylindrical shell to a depth of H).

This problem admits yet another class of solutions. Let us take the potential of velocities of the particles of fluid as

$$\Phi_{nm} = A_{nm} J_n (\varkappa_{nm}\xi) \operatorname{ch} \varkappa_{nm}\theta \cos n\varphi \sin \omega_{nm} t$$

where $J_n(x)$ is Bessel's function of real argument. In this form Φ_{nm} is a regular function of the volume of fluid with a rigid bottom and $\theta = 0$. Then instead of (1.9) and (1.11) we obtain

$$(2Eh / \rho_* gR^2) [\varkappa_{nm} J_{n-1}(\varkappa_{nm}) - nJ_n(\varkappa_{nm})] \varkappa_{nm}^3 - (n^2 - \varkappa_{mn}^2)^2 J_n(\varkappa_{nm}) \operatorname{th}(\varkappa_{nm} H / R) = 0 \tag{1.13}$$

$$\omega_{mn}^2 = (g\varkappa_{nm} / R) \operatorname{th}(\varkappa_{nm} H / R)$$

If we assume

$$A_{nm} = \frac{\omega_{nm} R (\varkappa_{nm}^2 - n^2)^2}{\varkappa_{nm} J_{n-1}(\varkappa_{nm}) - nJ_n(\varkappa_{nm})} C_{nm} \tag{1.14}$$

we will find

$$\begin{aligned} u_{nm} &= (n^2 + \sigma\varkappa_{nm}^2) \varkappa_{nm} C_{nm} \operatorname{sh} \varkappa_{nm}\theta \cos n\varphi \cos \omega_{nm} t \\ v_{nm} &= [n^2 - (2 + \sigma) \varkappa_{nm}^2] nC_{nm} \operatorname{ch} \varkappa_{nm}\theta \sin n\varphi \cos \omega_{nm} t \\ w_{nm} &= (n^2 - \varkappa_{nm}^2)^2 C_{nm} \operatorname{ch} \varkappa_{nm}\theta \cos n\varphi \cos \omega_{nm} t \end{aligned} \tag{1.15}$$

Usually $2Eh / \rho_* gR^2 \gg 1$. Then from (1.13) we conclude that the values $\varkappa_{nm}$ are approximately equal to the roots of the equation

$$J_{n-1}(x) - \frac{n}{x} J_n(x) = \frac{dJ_n(x)}{dx} = 0$$

which is identical with a known equation dealing with the free oscillation of fluid in a rigid cylindrical vessel [2]. Let us note that in this case and in agreement with (1.14) and (1.15) the velocities of the shell would be considered significantly smaller than the velocities of the particles of fluid.

2. Let us attempt now the problem of free oscillation of an elastic spherical shell, fully filled with fluid.

Equations of motion of the shell, with the usual symbols, have the form

$$\begin{gathered} \frac{\partial}{\partial\theta} \frac{1}{\sin\theta} \left(\frac{\partial u \sin\theta}{\partial\theta} + \frac{\partial v}{\partial\varphi} \right) - (1 + \sigma) \frac{\partial w}{\partial\theta} + (1 - \sigma) \left[u + \frac{1}{2\sin\theta} \frac{\partial}{\partial\varphi} \frac{1}{\sin\theta} \times \right. \\ \left. \times \left(\frac{\partial u}{\partial\varphi} - \frac{\partial v \sin\theta}{\partial\theta} \right) \right] - \frac{(1 - \sigma^2)\rho R^2}{E} \frac{\partial^2 u}{\partial t^2} = 0 \\ \frac{1}{\sin\theta} \frac{\partial}{\partial\varphi} \frac{1}{\sin\theta} \left(\frac{\partial u \sin\theta}{\partial\theta} + \frac{\partial v}{\partial\varphi} \right) - \frac{1 + \sigma}{\sin\theta} \frac{\partial w}{\partial\varphi} + (1 - \sigma) \left[v + \frac{1}{2} \frac{\partial}{\partial\theta} \frac{1}{\sin\theta} \times \right. \\ \left. \times \left(\frac{\partial v \sin\theta}{\partial\theta} - \frac{\partial u}{\partial\varphi} \right) \right] - \frac{(1 - \sigma^2)\rho R^2}{E} \frac{\partial^2 v}{\partial t^2} = 0 \\ \frac{1 + \sigma}{\sin\theta} \left(\frac{\partial u \sin\theta}{\partial\theta} + \frac{\partial v}{\partial\varphi} \right) - 2(1 + \sigma) w - \frac{(1 - \sigma^2)\rho R^2}{E} \frac{\partial^2 w}{\partial} - \frac{(1 - \sigma^2) R^2}{2Eh} \times \\ \times (p)_{\xi=1} = 0 \end{gathered} \tag{2.1}$$

1. Let the closed shell be free of any fixation and fully filled with fluid and perform its steady oscillations. The potential of the fluid in the spherical system of coordinates is taken as

$$\Phi_{nm} = B_{nm} \xi^m P_m{}^n (\cos\theta) \cos n\varphi \sin \omega_{nm} t \qquad (n = 0, 1, 2, \ldots) \tag{2.2}$$

where $P_m{}^n(x)$ are the associated functions of Legendre's polynomials. Then the solution of system (2.1), bearing in mind (1.4) and (1.5), is

$$u_{nm} = (1+\sigma) C_{nm} \frac{dP_m{}^n(\cos\theta)}{d\theta} \cos n\varphi \cos \omega_{nm} t \qquad (2.3)$$

$$v_{nm} = -(1+\sigma) n C_{nm} \frac{P_m{}^n(\cos\theta)}{\sin\theta} \sin n\varphi \cos\omega_{nm} t$$

$$w_{nm} = [1-\sigma-m(m+1)+(1-\sigma^2)\lambda_{nm}^2] C_{nm} P_m{}^n(\cos\theta) \quad \cos n\varphi \cos\omega_{nm} t$$

where $m = n + k$ ($k = 0, 1, 2, \ldots$). For this the parameter of frequency is determined by the equation

$$(1-\sigma^2)(1+cm)\lambda_{nm}^4 - [m^2+m-1+\sigma+(m^2+m+1+3\sigma)cm]\lambda_{nm}^2 + (m^2+m-2)\times$$
$$\times cm = 0 \qquad (2.4)$$

The constant in (2.2) was taken as

$$B_{nm} = [1-\sigma-m(m+1)+(1-\sigma^2)\lambda_{nm}^2] R\omega_{nm} C_{nm} / m$$

In the case when the forces of inertia of the shell are disregarded, we get

$$\lambda_{nm}^2 = \frac{[m(m+1)-2]cm}{m(m+1)-(1-\sigma)} \qquad (2.5)$$

One of the values of λ_{mn}^2, which is obtained from (2.4) will be too great and should be rejected on the grounds shown in Sect. 1.

If we neglect the inertia of the shell and take a sufficiently large value of c we can get excessive values of frequency, and with increase of the frequency of free oscillation this error becomes greater. Thus, in the case when $c = 0.078$, the parameter of frequency λ^2, calculated according to (2.5) for $m = 3, 5, 7, 9$, becomes, respectively, greater by 27, 41, 57, and 71% than its value determined by the quadratic equation (2.4).

If parities of n and m coincide, the modes of free oscillations are symmetrical with respect to the equatorial section of the system $\theta = \pi/2$. Otherwise they will be antisymmetric with respect to this section. The case of $m = 1$ corresponds to displacement of the system as a rigid body ($\lambda = 0$).

In paper [3] the problem of axisymmetrical oscillations is considered ($n = 0$) for an elastic closed spherical shell, partly filled with ideal incompressible fluid. Along the equator the shell is fixed against tangential displacements ($u = 0$) and inertia is disregarded. A separate case is considered when the shell is fully filled with fluid. In this latter case and in the case of the closed spherical shell without fixation with oscillations symmetrical relative to the equatorial section the solutions must be identical. This is confirmed by comparison of our results (disregarding the forces of inertia of the shell) with the results of paper [3].

2. The solution obtained for the closed spherical shell with antisymmetrical oscillations is also the solution for the problem of free oscillations of the half-spherical shell, fully filled with fluid with the boundary conditions

$$v = T_1 = p = 0 \qquad \text{for } \theta = \pi/2$$

If the last of these conditions is replaced by

$$R(\partial^2\Phi/\partial t^2) + g(\partial\Phi/\partial\theta) = 0 \qquad \text{for } \theta = \pi/2$$

we obtain

$$\lambda_{nm}^2 P_m{}^n(\cos\theta) - \delta\frac{dP_m{}^n(\cos\theta)}{d\theta} = 0 \quad \text{for } \theta = \pi/2$$

which together with (2.4) allows determination of the value m and λ_{nm}^2 (m is not an integer and $P_m{}^n(x)$ is the associated function of Legendre). For $\delta \lll \lambda^2$ approximate values of m are $m = n + k$ ($k = 1, 3, 5,.$)and therefore the application of the two different kinds of conditions to the free surface of fluid gives practically the same results.

BIBLIOGRAPHY

1. Pshenichnov G.I., Application of asymptotic method of integration to the problem of free oscillation of a thin elastic shell of revolution partly filled with fluid. Tr. VII All Union conference on the theory of shells and plates (Dnepropetrovsk, 1969), M., "Nauka", 1970.
2. Lamb G., Hydrodynamics. M.-L., Gostekhizdat, 1947.
3. Balabukh L.I., and Molchanov A.G., Axially symmetrical oscillations of spherical shell partly filled with fluid. Inzh. Zh. MTT, No. 5, 1967.

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THE RELATION BETWEEN MATHEMATICAL EXPECTATIONS OF STRESS AND STRAIN TENSORS IN ELASTIC MICROHETEROGENEOUS MEDIA

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Microheterogeneous media (composite materials, polycrystals and others) are examined for which the elastic moduli tensor c_{ijmn} is considered a homogeneous random function of coordinates. The question of the relation between mathematical expectations of stresses $\langle\sigma_{ij}\rangle$ and strains $\langle\varepsilon_{ij}\rangle$ in such media was studied by a number of authors [1-5] under the condition that the fields of stresses and strains are statistically homogeneous. The author of [6] examined the case of inhomogeneous fields and proposed a method of solution for the inhomogeneous stochastic problem. In this paper the program outlined in [6] is carried out.

In Sect. 1 the initial stochastic inhomogeneous problem is reduced to an infinite sequence of homogeneous problems. This is achieved through the representation of the solution in the form of a series which satisfies the equilibrium equations for a volume element of the body, and the equations of compatibility of deformations. The coefficients of this series are homogeneous random tensor functions which are independent of body form and also independent of the determined external load acting on the body. These tensor functions depend only on the elastic properties of the body and are completely determined through the given random tensor c_{ijmn}.

In Sect. 2 the coefficients of the above mentioned series are expressed in terms